Euler Solutions using Flux-based Wave Decomposition

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The Euler equations are solved for non-hydrostatic atmospheric flow problems in two dimensions using a high-resolution Godunov-type scheme. The Riemann problem is solved using a flux-based wave decomposition suggested by LeVeque. This paper describes in detail, the design and implementation of the Riemann solver used for computing the Godunov fluxes. The methodology is then validated against benchmark cases for non-hydrostatic atmospheric flows. Comparisons are made with solutions obtained from the National Center for Atmospheric Research's state-of-the-art numerical model. The method shows promise in simulating non-hydrostatic flows which, are characterized by steep gradients on the meso-, micro-, and urban-scales.

I. Introduction

Godunov-type methods (Godunov 1959, van Leer 1979) have gained wide popularity in the scientific computing community for solving the systems of hyperbolic conservation laws. Godunov's unique approach to numerical modeling of fluid flow is characterized by introducing physical reasoning in the development of the numerical scheme (van Leer 1999). The construction of the scheme itself is based upon the physical phenomenon described by the equation sets. These finite volume discretizations are conservative and have the ability to resolve regions of steep gradients accurately, thus avoiding dispersion errors in the solution. Positivity of scalars (an important factor when considering the transport of microphysical quantities) is also guaranteed by applying the *total variation diminishing* (TVD) condition appropriately.

In recent years there has been a growing interest in using Godunov-type methods for atmospheric flow problems. Ahmad et al. (2005) have implemented a Godunov-type scheme with the Harten-Lax-van Leer-Contact (HLLC) approximate Riemann solver (Toro et al. 1994) for the Euler as well as Navier-Stokes equations on unstructured adaptive grids. Botta et al. (2004) have simulated gravity waves forming over mountains. Carpenter et al. (1990) have applied the method for atmospheric flows using an exact Riemann solver in conjunction with the Piecewise Parabolic Method (Colella and Woodward 1984). Carpenter et al. (1990) show the inherent strengths of Godunov-type methods by providing a comparison with the Multidimensional Positive Definite Advection Transport Algorithm – MPDATA (Smolarkiewicz 1984) and the Leapfrog scheme. The important role Godunov-type methods can play in accurately resolving atmospheric phenomena characterized by steep gradients is also pointed out.

Hurricanes, tornadoes, fronts, drylines, micro-bursts and inversions are some examples of atmospheric processes, which have strong gradients of velocities, temperature and potential temperature. Fronts, for example, have large horizontal temperature and wind gradients and vertical wind shear. Strong convection in supercell thunderstorms can produce tornadoes, winds up to ~50 ms^{-1}, lightning and flash floods (Brooks and Doswell 1993). Drylines (Shaw et al. 1997) are characterized by a strong moisture gradient in the planetary boundary layer (e.g., in the Great Plains this gradient can be up to several degrees Celsius which is much larger than the climatological average of 0.04 $°Ckm^{-1}$ in the dewpoint temperature). Drylines can trigger strong convective activity and winds in excess of 50 miles per hour have been observed (Shaw et al. 1997). In tornadoes (F5 category), winds between 125 and 140 ms^{-1} have been observed. Hurricanes are yet another example of an atmospheric process which is characterized by extreme gradients of velocities and potential temperature (Gopalakrishnan et al. 2002).

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Traditionally, the centered finite difference schemes such as the Leapfrog scheme have been favored for discretizing the atmospheric flow equation set. These types of schemes exhibit dispersion errors (non-physical spurious oscillations), which can contaminate the numerical results (Carpenter et al. 1990). At smaller spatial and temporal scales (e.g., non-hydrostatic meso-scale flows), large gradients of velocities and other physical quantities can develop, which, create stability problems for centered schemes. The application of the Leapfrog scheme on smaller scales also requires explicit time filtering for stability. The Asselin time filter, which is often used, degrades the accuracy of the scheme in time (Durran 1991; Mendez-Nuñez and Carroll 1993). Furthermore, the scheme can introduce false negatives in important scalar microphysical quantities. To avoid false negatives either positive definite schemes (Smolarkiewicz 1984; Bott 1989) or Flux Corrected Transport (FCT)-type schemes (Boris and Book 1973; Zalesak 1979) are often used to advect scalar quantities, while the flow solver uses centered finite differences. The upwind-biased finite difference flow solvers, which have been implemented in atmospheric models, normally do not enforce the TVD condition (Klemp et al. 2000). The one notable exception is the MPDATA scheme (Smolarkiewicz 1984; Bacon et al. 2000; Smolarkiewicz 2006; Ahmad et al. 2006), which is sign-preserving in its basic form and can be made TVD by the appropriate use of limiters.

In this study a Godunov-type scheme suggested by LeVeque for hyperbolic conservation laws with spatially-varying flux functions (LeVeque 2002; Bale et al. 2002) is adapted for atmospheric flow problems. The scheme employs flux-based wave decompositions (*f-waves*) for the solution of Riemann problem and does not require the explicit definition of the Roe matrix (Roe 1981) to enforce conservation. This is an important property in the context of atmospheric flows since the Roe matrix for hyperbolic conservation laws governing atmospheric flows is not readily available. The other important feature of the scheme is its ability to incorporate source term due to gravity without introducing discretization errors. Again, in the context of atmospheric flows this is an important advantage. In the following sections the development of the Riemann solver and its implementation within the Conservation Laws Package (CLAWPACK) is described in detail. The numerical scheme is evaluated against benchmark test cases and a comparison is made with the results obtained from the National Center for Atmospheric Research's state-of-the-art numerical model (the Weather Research and Forecast model – also known as the ARW-WRF model).

II. Governing Equations

The basic equations governing atmospheric flows comprise of a set of partial differential equations for the conservation of mass, the conservation of momentum, the conservation of energy and an equation of state to close the system. The 2D Navier-Stokes equations governing atmospheric flows are written in the conservative form (e.g., see Ooyama 1990; Bacon et al. 2000; Koračin et al. 1998; Klemp et al. 2000) as:

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=Q+D \tag{1}$$

where,

$$U=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho\theta\end{bmatrix},\qquad F=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ u\rho\theta\end{bmatrix},\qquad G=\begin{bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ v\rho\theta\end{bmatrix} \tag{2}$$

In the above relations, ρ is the density of fluid, u is the velocity component in the x-direction, v is the velocity component in the y-direction and p is the pressure. If a parcel of air at temperature T and pressure p is subjected to an adiabatic compression or expansion to a final reference base-state pressure (p_0 = 10^5 Pa), then its potential temperature, θ is given by:

$$\theta=T\left(\frac{p_0}{p}\right)^{\frac{R_d}{c_p}} \tag{3}$$

The system is closed by an equation of state for pressure,

$$p = C_0 (\rho\theta)^\gamma \tag{4}$$

where C_0 is a constant given by:

$$C_0 = \frac{R_d^\gamma}{p_0^{R_d / C_v}} . \tag{5}$$

In the above relations, γ (= C_p/C_v = 1.4) is the ratio of specific heats, R_d (= 287 J K^{-1} kg^{-1}) is the gas constant for dry air, p_0 is the base state pressure (= 10^5 Pa). C_p (= 1004 J K^{-1} kg^{-1}) and C_v (= 717 J K^{-1} kg^{-1}) are the specific heats of air at constant pressure and volume respectively. In Eq. (1), Q is the source term and D is the diffusive flux term. The source term can be complex for atmospheric processes, and apart from body forces, may include terms for the heat sinks and sources produced due to the diurnal cycle of Earth, as well as microphysical processes of cloud formations and dissipations. For the purpose of this study, a simplified source term will be used. Atmosphere is assumed to be dry and the only source term is the gravitational force acting in the vertical direction. Diffusion processes are also ignored and only the Euler solutions are considered.

III. Numerical Scheme

The Riemann solver presented in this paper has been implemented within the Conservation Laws Package (CLAWPACK) software. CLAWPACK is a general purpose and open-source software developed at the University of Washington, Seattle, by LeVeque (2002). The software implements high-resolution Godunov-type methods which, are explained in detail in LeVeque (1996), LeVeque (1997), LeVeque (2002) and LeVeque (2003). The user has to provide the Riemann solver for the particular hyperbolic partial differential equation (there are several examples of Riemann solvers in the standard source distribution of CLAWPACK). The user also needs to provide the routines for model initialization and the addition of source terms, if any. The subroutines for dimensional-splitting, extensions for second order accuracy and various limiter functions, etc., are provided in CLAWPACK. The user can provide additional code for boundary conditions and add new limiter functions. In general, the software can be used to solve any hyperbolic partial differential equation set as long as its complete eigen-structure is known and the user provides the appropriate Riemann solver for it. Versions of CLAWPACK based on Message Passing Interface (MPI) for distributed computing and Adaptive Mesh Refinement (AMR) for computational efficiency, are also freely available.

In this section, a brief description of the base-line methodology implemented in CLAWPACK is given and the Riemann solver designed for the atmospheric flow equations is described in detail. The solution of the Riemann problem across each interface of a cell is a one-dimensional problem. Therefore, the basic algorithm is first described in one-dimension. The Euler equations (1)-(2) in one dimension can be written in the discrete form as:

$$U_i^{n+1} = U_i^n - \Delta t \left[\frac{1}{\Delta x} \left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right) \right], \tag{6}$$

where, U is the vector of conserved quantities, F is the vector of inter-cell fluxes calculated at the control surface of each control volume using either an exact or approximate Riemann solver. Δt and Δx are the time step and mesh resolution in x-direction respectively. In Godunov-type solvers, the jump in U at cell centers is usually decomposed as a linear combination of the eigenvectors, $r_{i-1/2}^p$ to obtain the waves, $W_{i-1/2}^p$:

$$U_i - U_{i-1} = \sum_{p=1}^{m} \alpha_{i-1/2}^p r_{i-1/2}^p \equiv \sum_{p=1}^{m} W_{i-1/2}^p . \tag{7}$$

The coefficients, $\alpha_{i-1/2}^p$, in Eq. (7) are given by:

$$\alpha_{i-1/2}^p = R_{i-1/2}^{-1} (U_i - U_{i-1}) , \tag{8}$$

where, $R_{i-1/2}$ is the matrix of right eigenvectors. The conservation is enforced by the following condition:

$$A_{i-1/2}(U_i - U_{i-1}) = F(U_i) - F_{i-1}(U_{i-1})\,, \tag{9}$$

where, $A_{i-1/2}$ is the Roe-averaged Jacobian (Roe 1981). Instead of solving the Riemann problem using the decomposition in Eq. (7), LeVeque (2002) and Bale et al. (2002) suggest using a flux-based wave decomposition, in which the flux differences $F_i(U_i) - F_{i-1}(U_{i-1})$ are written directly as a linear combination of the right eigenvectors $r^p_{i-1/2}$,

$$F_i(U_i) - F_{i-1}(U_{i-1}) = \sum_{p=1}^{m} \beta^p_{i-1/2} r^p_{i-1/2} \equiv \sum_{p=1}^{m} Z^p_{i-1/2}\,, \tag{10}$$

and,

$$\beta_{i-1/2} = R^{-1}_{i-1/2}(F_i(U_i) - F_{i-1}(U_{i-1})). \tag{11}$$

The vectors $Z^p = \beta^p r^p$ are called *f*-waves and contain flux increments rather than increments in *U*. In the presence of a source term, $\psi = \rho g$, the algorithm can be extended by first setting the source term values $\psi_{i-1/2}$ at the control surfaces and then defining the *f*-waves $Z^p_{i-1/2}$ as follows:

$$F_i(U_i) - F_{i-1}(U_{i-1}) - \Delta x \psi_{i-1/2} = \sum_{p=1}^{m} \beta^p_{i-1/2} r^p_{i-1/2} \equiv \sum_{p=1}^{m} Z^p_{i-1/2} \tag{12}$$

$$\beta_{i-1/2} = R^{-1}_{i-1/2}(F_i(U_i) - F_{i-1}(U_{i-1}) - \Delta x \psi_{i-1/2}). \tag{13}$$

The correction for source term due to gravity in Eq. (12)-(13) is applied only at the interior edges/faces. Eq. (6) can now be re-written as:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[\mathrm{H}^+\Delta U_{i-1/2} + \mathrm{H}^-\Delta U_{i+1/2}\right], \tag{14}$$

where,

$$\mathrm{H}^-\Delta U_{i-1/2} = \sum_p Z^p_{i-1/2} \quad if \quad s^p_{i-1/2} < 0\,, \tag{15}$$

and,

$$\mathrm{H}^+\Delta U_{i-1/2} = \sum_p Z^p_{i-1/2} \quad if \quad s^p_{i-1/2} > 0\,. \tag{16}$$

$\mathrm{H}^-\Delta U_{i-1/2}$ and $\mathrm{H}^+\Delta U_{i-1/2}$ are the fluctuations which contribute to the cell-averaged quantity U_i due to the wave propagation across the cell interfaces. In the above relations, $s^p_{i-1/2}$ are the wave speeds given by the eigenvalues of the hyperbolic equation set. Higher-order accuracy in space can be achieved by adding a correction term (LeVeque 2002; LeVeque 2003):

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[\mathrm{H}^+\Delta U_{i-1/2} + \mathrm{H}^-\Delta U_{i+1/2}\right] - \frac{\Delta t}{\Delta x}\left[\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right] \tag{17}$$

where,

$$\widetilde{F}_{1-1/2} = \frac{1}{2}\sum_{p=1}^{m} \text{sgn}(s^p_{i-1/2})\left[1 - \frac{\Delta t}{\Delta x}\left|s^p_{i-1/2}\right|\right]\widetilde{Z}_{i-1/2} \tag{18}$$

and, $\widetilde{Z}^p$ is the limited value of Z^p. Bale et al. (2002) have shown the above algorithm to be second-order accurate in space and time. The details on the implementation of wave limiters can be found in LeVeque (2002) and LeVeque (2003). Apart from the advantages of using Godunov-type schemes, mentioned earlier (conservation, ability to simulate regions of sharp gradients, etc.), the other important features of LeVeque's algorithm in the context of atmospheric flows are:

- The ease with which source term due to gravity can be included without introducing discretization errors.
- The method is attractive since the Roe matrix for the system of equations in Eq. (1)-(5) is not readily available.

In recent years there has been a growing interest in developing multi-dimensional Riemann solvers (e.g., see Brio et al. 2001). This research however, is still in its infancy and most Godunov-type codes for multi-dimensional problems are based on the solution of the one-dimensional Riemann problem across each cell interface. It has been shown (LeVeque 1996; LeVeque 1997; Langseth and LeVeque 2000) that the multi-dimensional information can be included into the solution even though the calculation of the Riemann problem is only one-dimensional. The simplest way to extend the algorithm for multi-dimensional problems is by the dimensional-splitting technique in which, individual sweeps are performed in each spatial direction in succession. A better algorithm (LeVeque 1996) takes into account contributions of wave fluctuations in the transverse direction.

A. Eigenvalues and Right Eigenvectors of the Euler Equations Governing Atmospheric Flows

In the Riemann solution suggested by LeVeque, it is assumed that the right eigenvectors and eigenvalues of the conservation laws can be obtained from an approximate Jacobian matrix. The eigenspace for the system of hyperbolic equations governing atmospheric flows in two-dimensions is given in this section. Consider the homogeneous part of Eq. (1)-(2), then the Jacobian *A(U)* corresponding to *F(U)* in Eq. (1) can be written as:

$$A(U) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} & \frac{\partial f_1}{\partial u_4} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} & \frac{\partial f_2}{\partial u_4} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial u_3} & \frac{\partial f_3}{\partial u_4} \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} & \frac{\partial f_4}{\partial u_3} & \frac{\partial f_4}{\partial u_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -u^2 & 2u & 0 & a^2/\theta \\ -uv & v & u & 0 \\ -u\theta & \theta & 0 & u \end{bmatrix}, \tag{19}$$

where,

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho\theta \end{bmatrix} \quad and, \quad F(U) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} u_2 \\ \frac{u_2^2}{u_1} + C_o u_4^\gamma \\ \frac{u_2 u_3}{u_1} \\ \frac{u_2 u_4}{u_1} \end{bmatrix}. \tag{20}$$

The eigenvalues of the matrix *A(U)* in Eq. (19) are *u*, *u*, *u+a*, and *u-a*, where, *a* is the speed of sound. The right eigenvectors of the matrix are as follows:

$$u-a:\begin{bmatrix}1\\u-a\\v\\\theta\end{bmatrix}\quad u:\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\quad u:\begin{bmatrix}1\\u\\0\\0\end{bmatrix}\quad u+a:\begin{bmatrix}1\\u+a\\v\\\theta\end{bmatrix},\tag{21}$$

Similarly, the Jacobian, *A(U)* corresponding to *G(U)* in Eq. (1) is given by:

$$A(U)=\begin{bmatrix}0 & 1 & 0 & 0\\-uv & u & v & 0\\-v^2 & 2v & 0 & a^2/\theta\\-v\theta & \theta & 0 & v\end{bmatrix}\tag{22}$$

The eigenvalues and the corresponding right eigenvectors of the Jacobian in Eq. (22) are:

$$v-a:\begin{bmatrix}1\\u\\v-a\\\theta\end{bmatrix}\quad v:\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\quad v:\begin{bmatrix}1\\0\\v\\0\end{bmatrix}\quad v+a:\begin{bmatrix}1\\u\\v+a\\\theta\end{bmatrix}.\tag{23}$$

B. The Riemann Solver in the Normal Direction

Consider the matrix of right eigenvectors *R* (each column is a right eigenvector), for *A(U)* corresponding to *F(U)* in Eq. (19), then the matrix *R* and its inverse R^{-1} (the inverse matrix R^{-1}, is the matrix of left eigenvectors in which each row is a left eigenvector) can be written as:

$$R=\begin{bmatrix}1 & 0 & 1 & 1\\u-a & 0 & u & u+a\\v & 1 & 0 & v\\\theta & 0 & 0 & \theta\end{bmatrix}\quad;\quad R^{-1}=\begin{bmatrix}\frac{u}{2a} & -\frac{1}{2a} & 0 & \frac{1}{2\theta}\\0 & 0 & 1 & -\frac{v}{\theta}\\1 & 0 & 0 & -\frac{1}{\theta}\\-\frac{u}{2a} & \frac{1}{2a} & 0 & \frac{1}{2\theta}\end{bmatrix}.\tag{24}$$

The β_i coefficients can now be calculated using Eq. (11) by simply multiplying the inverse of the matrix of right eigenvectors with the vector containing the jumps in fluxes:

$$\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\\\beta_4\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\left(\frac{u}{a}\Delta F_1-\frac{1}{a}\Delta F_2+\frac{1}{\theta}\Delta F_4\right)\\\Delta F_3-\frac{v}{\theta}\Delta F_4\\\Delta F_1-\frac{1}{\theta}\Delta F_4\\\frac{1}{2}\left(-\frac{u}{a}\Delta F_1+\frac{1}{a}\Delta F_2+\frac{1}{\theta}\Delta F_4\right)\end{bmatrix},\tag{25}$$

where, the jumps, ΔF_i in the flux functions are given by:

$$\begin{bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \\ \Delta F_4 \end{bmatrix} = \begin{bmatrix} (\rho u^j)_i - (\rho u^j)_{i-1} \\ (\rho u^j u^j)_i + p_i - (\rho u^j u^j) - p_{i-1} \\ (\rho u^j v^j)_i - (\rho u^j v^j)_{i-1} \\ (\rho \theta u^j)_i - (\rho \theta u^j)_{i-1} \end{bmatrix}. \tag{26}$$

The superscript j in Eq. (26) implies the component of the momentum equation (in x- or y-direction) and the subscript i denotes the cell number. The source term due to gravity is added when the Riemann problem is solved for the sweep in the y-direction:

$$\Delta F_2 = \Delta F_2 + 0.5 \Delta y g \left[\rho_{i-1}^r + \rho_i^l \right], \tag{27}$$

where, g is the acceleration due to gravity. The pressure in Eq. (26) is calculated using the equation of state given by Eq. (4)-(5). Once the β_i coefficients have been calculated, the *f-waves*, $Z^p = \beta^p r^p$ can be computed as follows:

$$u-a: \begin{bmatrix} Z_1^1 \\ Z_2^1 \\ Z_3^1 \\ Z_4^1 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_1(u-a) \\ \beta_1 v \\ \beta_1 \theta \end{bmatrix} \quad ; \quad u: \begin{bmatrix} Z_1^2 \\ Z_2^2 \\ Z_3^2 \\ Z_4^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \beta_2 \\ 0 \end{bmatrix} \quad ; \quad u: \begin{bmatrix} Z_1^3 \\ Z_2^3 \\ Z_3^3 \\ Z_4^3 \end{bmatrix} = \begin{bmatrix} \beta_3 \\ \beta_3 u \\ 0 \\ 0 \end{bmatrix} \quad ; \quad u+a: \begin{bmatrix} Z_1^4 \\ Z_2^4 \\ Z_3^4 \\ Z_4^4 \end{bmatrix} = \begin{bmatrix} \beta_4 \\ \beta_4(u+a) \\ \beta_4 v \\ \beta_4 \theta \end{bmatrix}. \tag{28}$$

Given the *f-waves* and the wave speeds (u-a, u, u, u+a), the flux differences can be computed by summing up the left and right going waves across a cell interface using Eq. (15)-(16). In the above relations the sweep in x-direction is implied. Computations in the y-direction would require interchanging u and v momentums. The quantities on cell faces are calculated by taking the average of cell-centered quantities on the either side of the face.

C. The Riemann Solver in the Transverse Direction

The solution of Riemann problem in the transverse direction is analogous to the solution in the normal direction. The β_i coefficients are calculated by replacing in Eq. (25) the jumps in flux functions, ΔF_i by the fluctuations $\mathrm{H}^- \Delta U_{i-1/2}$ and $\mathrm{H}^+ \Delta U_{i-1/2}$. The waves can now be computed as follows:

$$v-a: \begin{bmatrix} Z_1^1 \\ Z_2^1 \\ Z_3^1 \\ Z_4^1 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_1 u \\ \beta_1(v-a) \\ \beta_1 \theta \end{bmatrix} \quad ; \quad v: \begin{bmatrix} Z_1^2 + Z_1^3 \\ Z_2^2 + Z_2^3 \\ Z_3^2 + Z_3^3 \\ Z_4^2 + Z_4^3 \end{bmatrix} = \begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_3 v \\ 0 \end{bmatrix} \quad ; \quad v+a: \begin{bmatrix} Z_1^4 \\ Z_2^4 \\ Z_3^4 \\ Z_4^4 \end{bmatrix} = \begin{bmatrix} \beta_4 \\ \beta_4 u \\ \beta_4(v+a) \\ \beta_4 \theta \end{bmatrix}, \tag{29}$$

where, all waves at speed v have been summed up, since only the fluctuations are required. In all of the computations presented below, the Riemann problem in both the normal and the transverse directions was solved. The solution of the Riemann problem in the transverse direction allows larger time steps and a CFL of 0.9 was used for all simulations. It also increases the accuracy of the solution by taking into account the contributions of each wave in the transverse direction. The use of the terms “normal direction” and “transverse direction” is with respect to the control surfaces.

D. Extension of the Solver to the Third Spatial Dimension

The Riemann solver presented here can be extended to the third spatial dimension, if the wave structure is known for the equation set in three dimensions. The complete eigenspace of the hyperbolic partial differential equations governing atmospheric flows in three dimensions is given in Appendix A. The implementation of the algorithm in 3D, using dimensional-splitting would be straightforward (in each step a sweep of calculations is performed in the x-direction and then a similar sweeps in y- and z-directions are applied). However, if the corrections in the transverse direction are taken into account, then the implementation may become involved and/or computationally expensive.

LeVeque (2003) notes, that for some three-dimensional problems, using dimensional-splitting may be more efficient computationally.

IV. Results

In this section the proposed scheme is evaluated against different benchmark cases. In the absence of closed form solutions for Eq. (1)-(5), the dynamical cores of atmospheric flow models are usually evaluated in qualitative terms. A series of test cases have been developed over the years, and the results of four of these tests are presented below – the propagation of inertia-gravity waves on non-hydrostatic scales, the convection of a warm bubble in neutral atmosphere, and two variations of non-linear density currents. To better gauge the performance of the Godunov-type scheme, the results of the first two test cases are compared with the solutions obtained from the National Center for Atmospheric Research's state-of-the art, WRF model.

A. Non-hydrostatic Inertia-Gravity Waves

The Skamarock-Klemp (1994) test for simulating inertia-gravity waves on the non-hydrostatic scale is described in this section. The domain was bounded within [0:300.0km] x [0:10.0km]. The model was tested for different mesh resolutions – varying from 1km to 500m in the horizontal and 100m to 50m in the vertical. Periodic boundary conditions were used in the lateral and the top and bottom boundaries were set to solid walls. The domain was initialized by a constant Brunt-Väisälä frequency $N = 10^{-2}s^{-1}$. The waves were excited by an initial potential temperature perturbation given by:

$$\theta(x, y, 0) = \Delta\theta_0 \frac{\sin(\pi y / H)}{1 + (x - x_c)^2 / a^2} \qquad (30)$$

The amplitude of the initial potential temperature perturbation, $\Delta\theta_0$ was set to 10^{-2}K. The height H of the domain was 10km, the perturbation half width was a = 5km. The perturbation was initialized at $x_c = x_{max}/3$, where, x_{max} is the width of the domain (300km). A uniform u-velocity of 20m/s was imposed in the domain and the v-velocity was set to zero. The model was run for time = 3000s.

The ARW-WRF (version 2.1.2) model (Klemp et al. 2000) was used to run the non-hydrostatic inertia gravity wave case for comparison with the proposed Godunov-type scheme. The WRF simulations were run using different orders of advection. The first simulation employed a 2nd order centered finite difference scheme in the horizontal and vertical, while the second simulation had 5th order horizontal and 3rd order vertical advection (generally recommended by the developers of WRF). Please note that the upwind-biased schemes in the WRF model do not enforce the TVD condition and therefore are susceptible to spurious oscillations. The solution is integrated in time using an explicit multi-stage Runge-Kutta time-marching scheme (Jameson et al. 1981; Wicker and Skamarock 1998). In the WRF runs the Coriolis parameter was set to zero. WRF's lower boundary was set to free-slip, and the upper boundary employed a vertical velocity = 0 at a constant pressure level. Periodic boundary conditions were set in the lateral. Horizontal grid spacing was 1000m, and the vertical grid spacing was set such that the interval was roughly 100m for the baseline case. The mesh resolution in the vertical was adjusted accordingly for the high-resolution run. The turbulent diffusion was turned off, and no wave-absorbing sponge layer was used in the upper part of the computational domain.

The potential temperature perturbation field and the vertical velocity for the Godunov-type scheme, after 3000s into the simulation are shown in Figure 1 and Figure 2 respectively. The mesh resolution in this simulation was set to 500m in the horizontal and 50m in the vertical. Please note that the simulation results are slightly different than the analytical solution given in Skamarock and Klemp (1994). This is because the analytical solution in Skamarock and Klemp (1994) assumes a Boussinesq atmosphere and the computed results presented here are obtained from a fully compressible flow model. However, the results compare well with the solutions from the WRF model which also computes the fully-compressible equation set given by Eq. (1)-(5). Figures (3)-(4) show the comparison of the proposed Godunov-type scheme with the WRF 2nd order centered and the 5th order upwind-biased schemes at time = 3000s. The solution obtained from the WRF 5th order scheme is considered as the reference solution and is given by the solid line in Figures (3)-(4) – the mesh resolution for the 5th order WRF reference solution was 1km in the horizontal and 100m in the vertical. The high-resolution Godunov-type scheme compares well with the reference solution as well as the WRF second-order scheme. Some numerical diffusion is present in the low-resolution run but that is overcome by increasing the mesh resolution. Furthermore it does not exhibit dispersion and phase errors apparent in the WRF 2nd order solution. For the Godunov-type scheme, it should be noted that if the vertical mesh resolution is too coarse (approximately > 200m), then errors start to accumulate at the lower and upper domain boundaries due to source term balancing. Since, the proposed scheme is meant for flows on high-resolution meshes (where centered finite difference schemes can break down) this would not be an issue of concern. It may be possible to avoid these errors by using a stretched grid (the vertical mesh resolution in the surface layer is typically ~20m for stretched grids) near both upper and lower domain boundaries, but that needs to be tested.

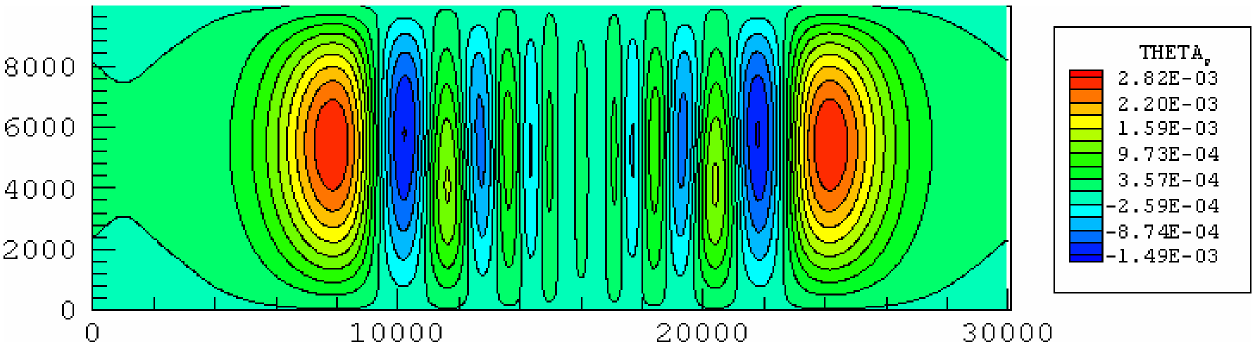

Figure 1. Non-hydrostatic Inertia-Gravity Waves. Potential temperature perturbation (K) field after 3000s into the simulation. Please note that the *x*-axis has been scaled by a factor of 10 for plotting purposes. The actual domain bounds in *x*-direction are [0:300km]. Δx=500m and Δy=50m. Godunov solution.

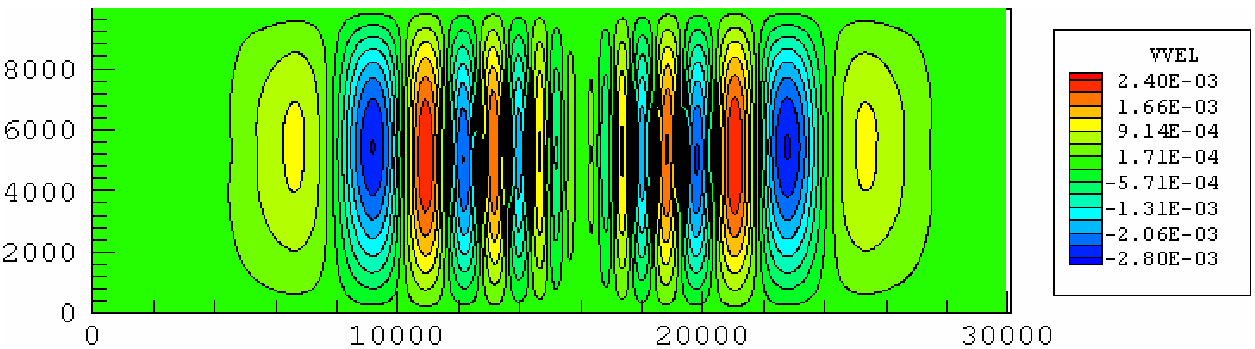

Figure 2. Non-hydrostatic Inertia-Gravity Waves. *v*-velocity (m/s) field after 3000s into the simulation. Please note that the *x*-axis has been scaled by a factor of 10 for plotting purposes. The actual domain bounds in *x*-direction are [0:300km]. Δx=500m and Δy=50m. Godunov solution.

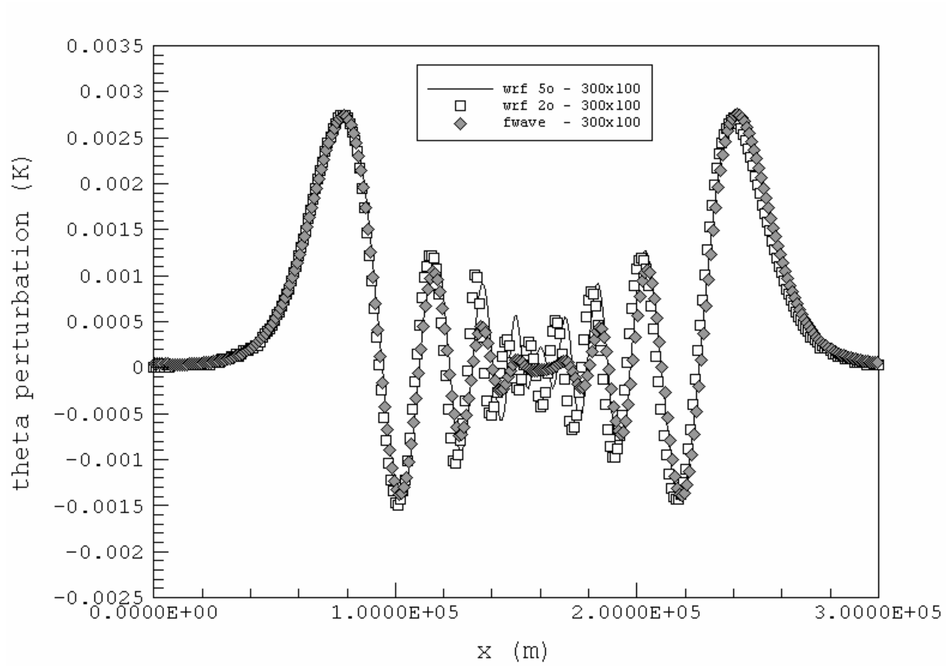

Figure 3. Non-hydrostatic Inertia-Gravity Waves. Comparison of the Godunov-type scheme with the 5th order and 2nd order WRF schemes. Theta perturbation (K) values between x = 0 & 300km, for y = 5km. 300x100 (Δx=1000m and Δy=100m) in the legend refer to number of cells in x and y-direction respectively.

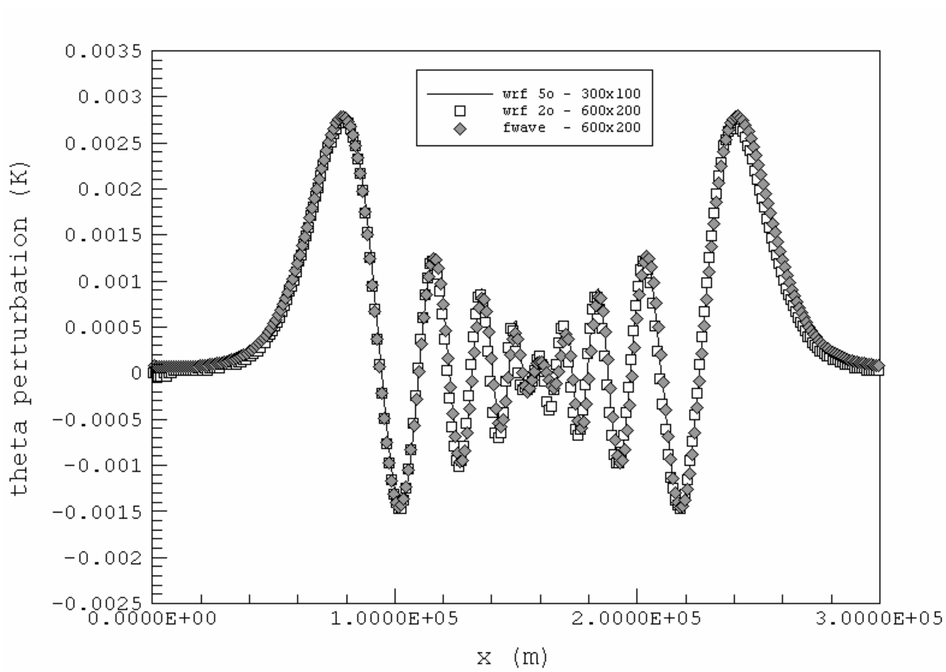

Figure 4. Non-hydrostatic Inertia-Gravity Waves. Comparison of the Godunov-type scheme with the 5th order and 2nd order WRF schemes. Theta perturbation (K) values between x = 0 & 300km, for y = 5km. 600x200 (Δx=500m and Δy=50m) and 300x100 (Δx=1000m and Δy=100m) in the legend refer to number of cells in x and y-direction respectively.

B. Convection in Neutral Atmosphere

The convection in neutral atmosphere is a popular test case for validating numerical schemes for atmospheric flow simulations (*e.g.*, Ahmad et al. 2005; Wicker and Skamarock 1998; Koračin et al. 1998; Mendez-Nuñez and Carroll 1994; Carpenter et al. 1990). In the absence of an analytical solution, the test can evaluate the scheme only in qualitative terms. This evaluation however, provides valuable information on the scheme's ability to simulate the fundamentals of atmospheric thermodynamics and dynamics.

The domain for this case was bounded within [0:20.0km] x [0:10.0km]. The mesh resolution was set to 125m. Farfield/outflow boundary conditions were used in the lateral and the top and bottom boundaries were set as solid walls. The domain was initialized for a neutral atmosphere at θ_i = 300 K in hydrostatic balance. The initial *u*-velocity and the *v*-velocity were set to zero. A potential temperature perturbation in the form of a linear function was initialized at $x_b = x_{max}/2$ and y_b = 2000m:

$$\theta(x,y) = \theta_i + \Delta\theta\left(1.0 - \frac{L}{R}\right) \quad if \quad L \le R \tag{31}$$

where,

$$L = \sqrt{(x - x_b)^2 + (y - y_b)^2} \tag{32}$$

The potential temperature perturbation maximum $\Delta\theta$, had a value of 2K. The radius *R*, of the perturbation was set to 2000m. The model was run for time = 1020s (17 minutes). The domain configuration is same as the one used in Wicker and Skamarock (1998) – the only difference is that final time of the simulation was increased to 1020 seconds (17 minutes) from 1000 seconds, since the WRF output is in multiples of minute. The *monotonized-centered* (MC) limiter was used to enforce the TVD condition. The initial potential temperature field and the simulation results after time = 1020s are shown in Figure 5 along with results from the WRF model.

The WRF setup for this case was similar to the inertia-gravity wave case described in the previous section, except for the lateral boundary conditions (outflow boundary conditions were specified instead of periodic). The horizontal and vertical mesh resolutions for the WRF model were set to 125m, but it should be noted that the vertical mesh resolution in meters is only an approximation – since, WRF uses a mass-based vertical coordinate system, the height of the vertical levels change with each time step. The time step in WRF simulations is a user input (rather than a calculation within the time loop) and it was set to 0.25 seconds. The WRF model employs an Arakawa-C staggered grid, where the geo-potential is collocated with the vertical velocity points. The geo-potential levels are used to obtain the Cartesian height. For post-processing of the data, the geo-potential and the velocity components are linearly interpolated to the mass coordinate points where the potential temperature is defined. Linear interpolation is then done to place these values on a new grid with constant height surfaces. One complication is the lowest surface on the new grid, which is 0m in the vertical direction. This is below the lowest potential temperature surface in the WRF output files. For this test case, the value of potential temperature at height = 0m is set to the potential temperature value at the lowest corresponding level in the WRF output.

The introduction of the thermal in the domain generates large acceleration in the center of the bubble accompanied by downdrafts on the either side of the bubble. Since, the temperature distribution inside the bubble is linear (highest temperature at the center) the center of the bubble rises faster. This creates sharp gradients of temperature in the upper part of the bubble (Koračin et al. 1998). As the bubble rises, the initial symmetry of the flow field is broken and an off-axis maximum of buoyancy is created (Carpenter et al. 1990). The results can be compared with, e.g., the Wicker and Skamarock (1998) solution, which exhibits spurious oscillations in the potential temperature field. These oscillations are typical of the centered finite difference schemes. The high-resolution Godunov solution presented here, on the other hand is devoid of these oscillations. The 2nd order WRF scheme became unstable for this test case and a constant diffusion (with K=15m^2/s) was added to stabilize the solution (shown in Figure 5). The vertical velocity acceleration and the thermal energy dissipation rates for the three models are quite similar and can be seen in the time histories plotted in Figure 6. WRF outputs data at 1 minute increments and therefore only 17 data points were available for WRF time histories. Figure 7 shows the velocity fields obtained from the Godunov scheme after time = 1020s into the simulation.

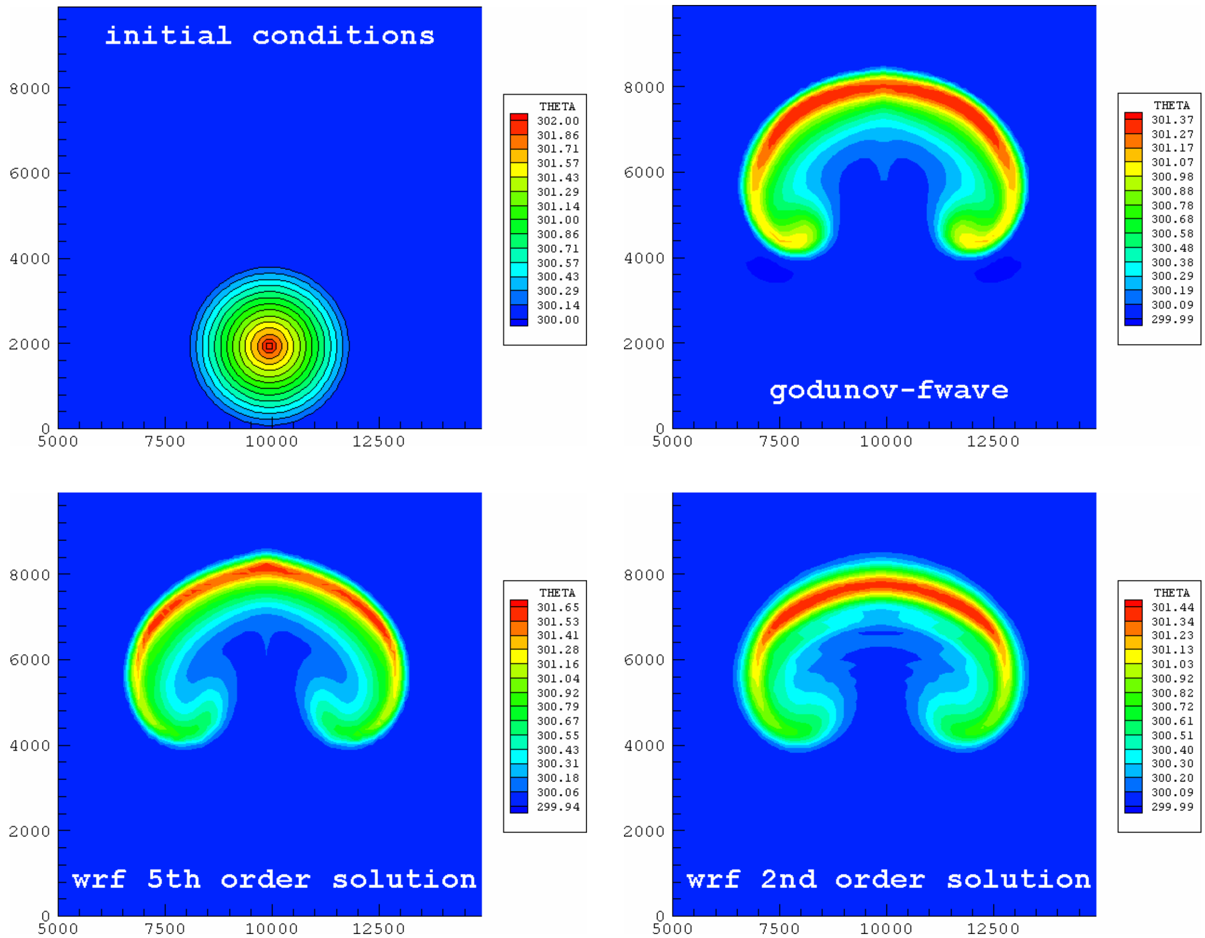

Figure 5. Convection in Neutral Atmosphere. Initial conditions (top-left). Godunov solution (top-right), WRF 5th order solution (bottom-left), and WRF 2nd order solution at time = 1020s (bottom-right).

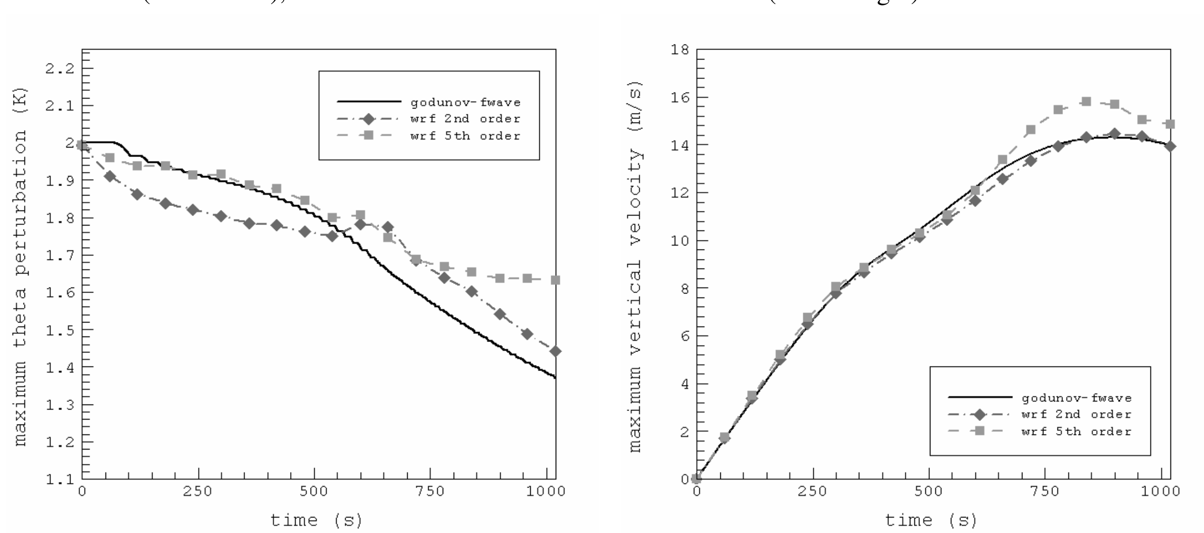

Figure 6. Convection in Neutral Atmosphere. Time history of domain maximum potential temperature perturbation (left) and the domain maximum vertical velocity (right) are shown in the figure.

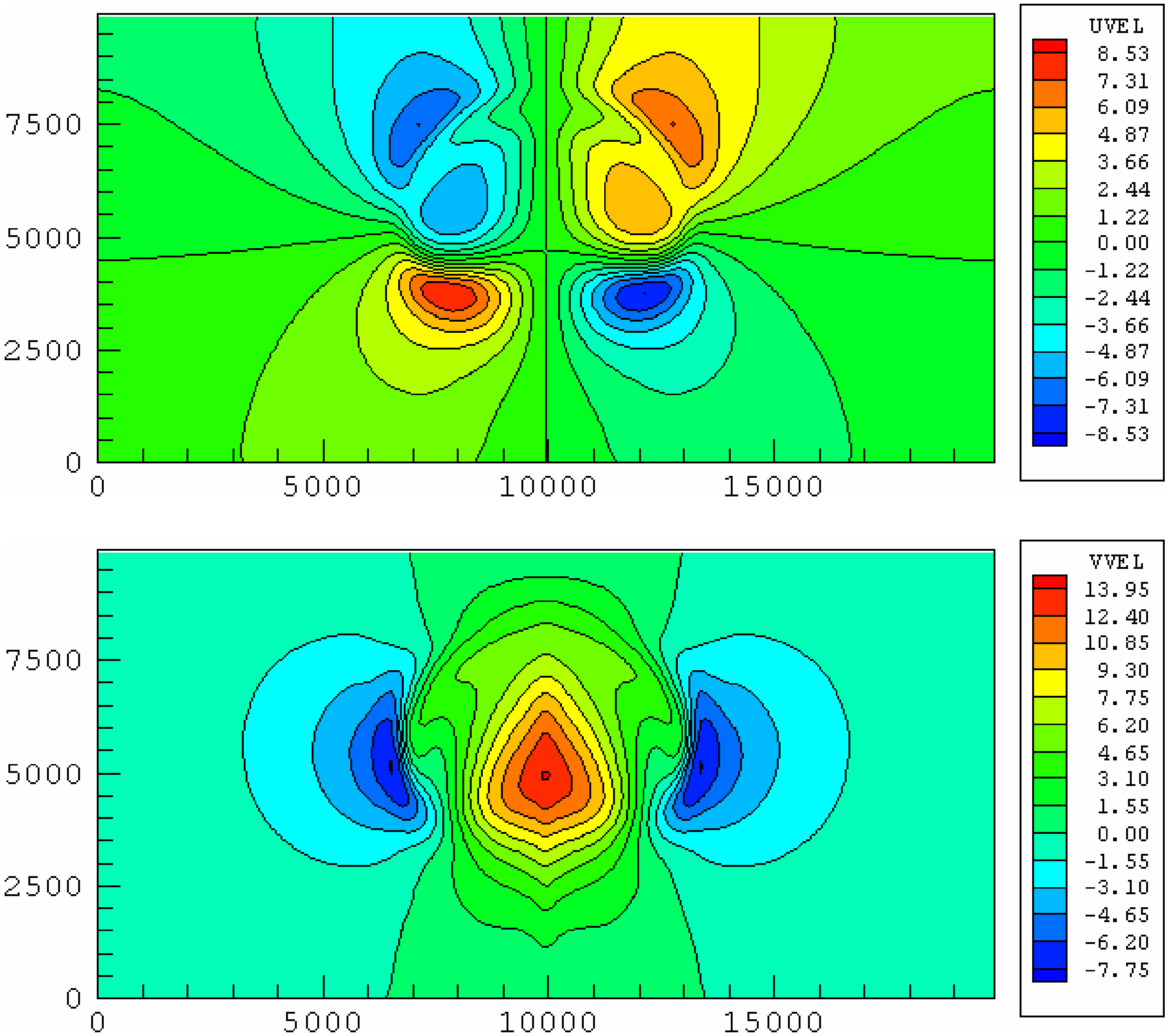

Figure 7. Convection in Neutral Atmosphere. *u*-velocity in m/s (top) and *v*-velocity in m/s (bottom) at simulation time = 1020s.

C. Modified Straka Density Current

The density current benchmark suggested by Straka et al. (1993) is often used to evaluate the dynamical cores of atmospheric flow models. In this section, a modified Straka (1993) benchmark is simulated with the following difference – since, only Euler solutions are considered in this study, the constant diffusion (with K=75m^2/s) is not added. Please see Appendix B for the actual Straka benchmark.

The domain in this case was bounded within [-26.5km:26.5km] x [0:6.4km]. The mesh resolution was set to 50m in both the *x*- and *y*-directions. Farfield/outflow boundary conditions were used in the lateral and the top and bottom boundaries were set as solid walls. The domain was initialized for a neutral atmosphere at 300 K in hydrostatic balance. The initial *u*-velocity and the *v*-velocity were set to zero. A cold bubble was initialized using the following relation:

$$\Delta\theta = \begin{cases} 0.0 & \text{if } L > 1.0, \\ -15.0\left[\dfrac{\cos(\pi L)+1.0}{2}\right] & \text{if } L \le 1.0 \end{cases} \tag{33}$$

where,

$$L = \sqrt{\left(\frac{x - x_c}{x_r}\right)^2 + \left(\frac{y - y_c}{y_r}\right)^2} \tag{34}$$

where, x_c = 0.0km, y_c = 3.0km, x_r = 4.0km and y_r = 2.0km. The simulation was run for time = 900s. The *monotonized-centered* (MC) limiter was used to enforce the TVD condition. The potential temperature fields at different simulation times (time = 300s, 600s and 900s) are shown in Figure 8. Only the right half of the computational domain is shown in the figures.

The test case can be considered as an idealized micro-burst. As the cold air descends due to negative buoyancy, strong downdrafts develop at the center of the cold bubble. When the cold air reaches the ground it is rolled up and a front is formed. As this front propagates, shear is generated at the top boundary of the front. The benchmark solution (Straka et al. 1993) consists of three rotors, which develop at the top boundary of the front due to the Kelvin-Helmholtz type instability. The formation and the propagation of the front and the development of these rotors can be seen in Figure 8. The final front location is also indicated in Figure 8, which compares well with the solutions given in Straka et al. (1993). The front location (in terms of potential temperature) at 14975m corresponds to the cell with a potential temperature value of 299.99 K – the value of potential temperature in the cell next to it (at 15025m) is 300.0K. The WRF 2nd order simulation (not shown here) became unstable for this test case also – but the addition of explicit diffusion stabilized the solution.

D. Droegemeier Density Current

The density current benchmark suggested by Straka et al. (1993) is based on a numerical experiment designed by Droegemeier (Droegemeier and Wilhelmson 1987; Carpenter et al. 1990). In this section the simulation of Droegemeier's experiment is described. The domain in this case was bounded within [0km:25.0km] x [0:10.0km]. The mesh resolution was set to 50m in both the *x*- and *y*-directions. Farfield/outflow boundary conditions were used in the right lateral boundary and the left lateral boundary was set to wall. The top and bottom boundaries were also set as solid walls. The domain was initialized for a neutral atmosphere at 300 K in hydrostatic balance. The initial *u*-velocity and the *v*-velocity were set to zero. A cold pool of air was initialized between [0:5000m] x [0:5000m], which linearly decreased in temperature to 290K at the surface (see Figure 9). The simulation was run for time = 900s. The *monotonized-centered* (MC) limiter was used to enforce the TVD condition.

Like in the previous density current case, the negative buoyancy sets the pool of cold air in a downward motion, and as this cold air descends, a front is formed which, propagates forward. This forward motion of the cold front generates shear at the top boundary of the front, resulting in Kelvin-Helmholtz waves. The propagation of front and the formation of these waves can be seen at time = 450s and time = 900s into the simulation (Figure 9). Please note that this is also an Euler solution and again no diffusion was added in the simulation. A qualitative comparison can be made with the solutions (PPM and centered finite difference scheme) given in Carpenter et al. (1990).

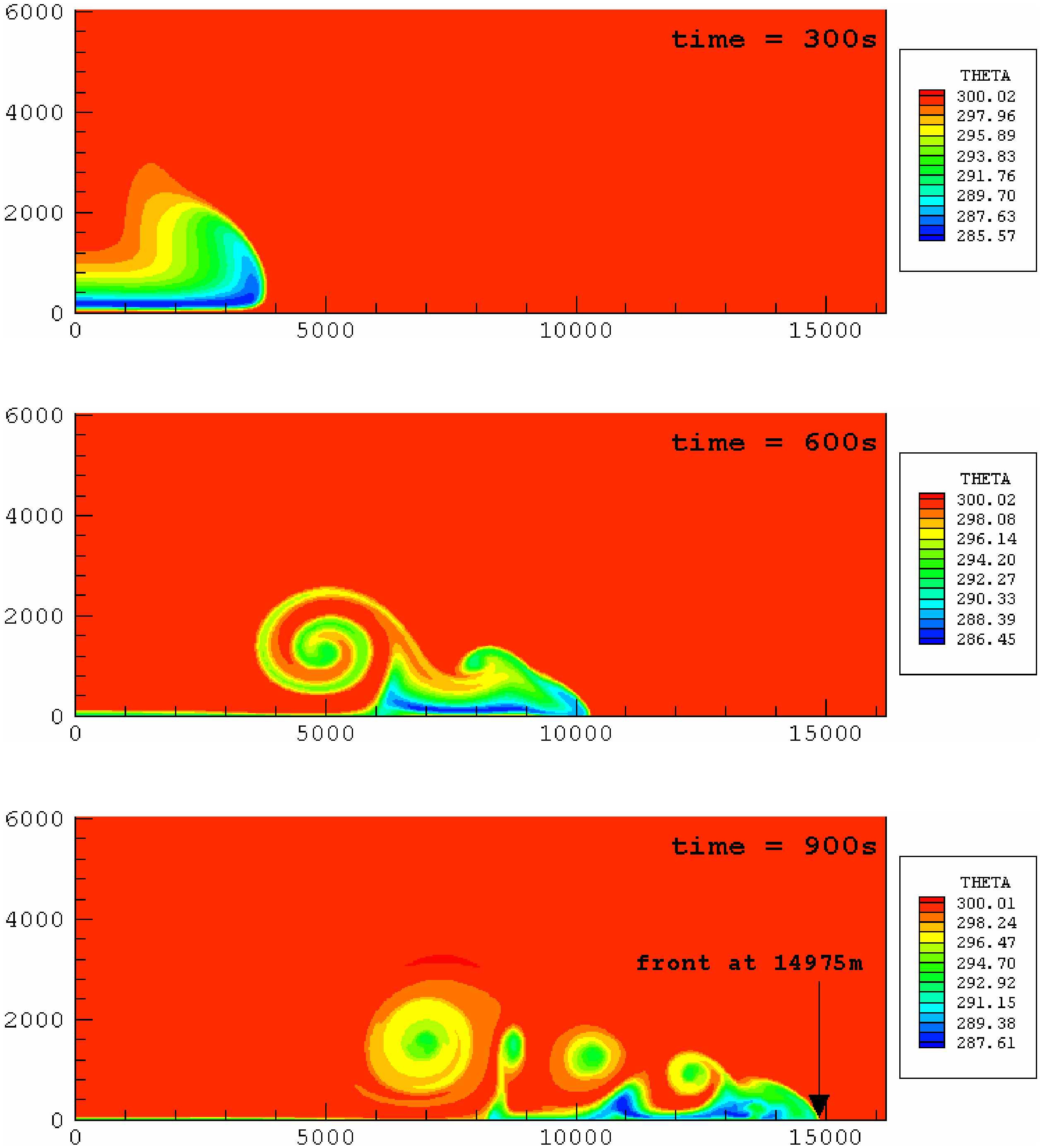

Figure 8. Density Current (after Straka et al.). Potential temperature (K) fields at time = 300s, 600s, and 900s are shown. The location of the front at time = 900s is also given. Only a part of the right half of the computational domain is shown in the figures. The formation of a front can be seen at time=300s. The propagation of this front generates large amount of shear in the flow. The development of rotors due to Kelvin-Helmholtz-type instability can be seen at time = 600s and time = 900s.

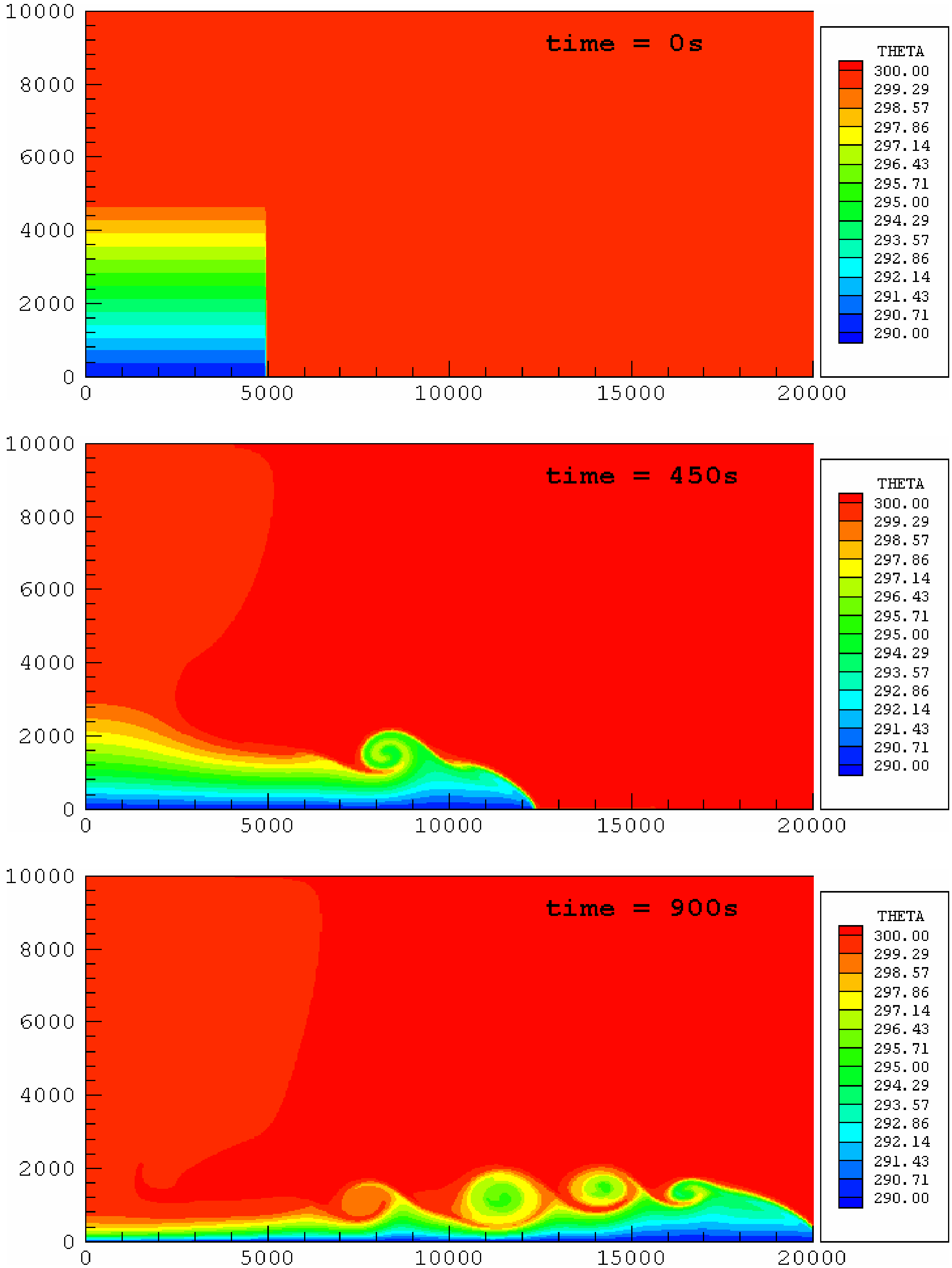

Figure 9. Density Current (after Droegemeier and Wilhelmson). Potential temperature (K) fields at time = 0, 450s, and 900s are shown. The actual domain extends to 25km in the *x*-direction.

V. Conclusions

A high-resolution Godunov-type scheme based on flux-based decomposition of waves after LeVeque is suggested for simulating atmospheric flow problems. Benchmark problems for the non-hydrostatic atmosphere were simulated for the validation of the scheme and the results are quite encouraging. Godunov-type high-resolution methods offer an attractive alternative for simulating atmospheric flows on the meso-, micro-, and urban-scales, which are characterized by steep gradients (micro-bursts, tornadoes, hurricanes, accidental or intentional release of hazardous materials in an urban area, etc.). These finite volume discretizations are conservative and have the ability to resolve regions of steep gradients accurately, thus avoiding dispersion errors in the solution. Numerical diffusion (implicit in all upwind schemes) can be overcome by higher-order extensions or using solution-adaptive methods. The positivity of scalars is also achieved by the appropriate use of limiters. A comparison with the National Center for Atmospheric Research's state-of-the-art WRF model showed that the high-resolution Godunov-type scheme performs well for flows in which sharp gradients can develop. For example, the WRF 2nd order scheme becomes unstable if large gradients are present in the solution (density current and convection in neutral atmosphere cases) and requires explicit filtering for stability. Although the study demonstrates the viability of using Godunov-type schemes for atmospheric flow simulations, much can be done to further improve the methodology presented here, such as, inclusion of more physics (turbulence closure, surface layer physics, microphysics, radiation transfer, etc.). The scheme can be implemented on unstructured grids and that is another avenue for future extensions.

Acknowledgments

Many thanks to Prof. Randall J. LeVeque for providing the CLAWPACK version which, implements the *f-waves* decomposition for solving the Riemann problem. This software was used as a template for implementing the Riemann solver for conservation laws governing atmospheric flows. We are grateful to Dr. R. J. Purser for reviewing the paper and for his helpful comments.

Appendix A

The eigen-structure for the three-dimensional Euler equations is given in this Appendix. The homogeneous Euler equations governing atmospheric flows can be written as:

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}=0 \tag{35}$$

where,

$$U=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho\theta\end{bmatrix},\quad F=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\\ u\rho\theta\end{bmatrix},\quad G=\begin{bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho vw\\ v\rho\theta\end{bmatrix},\quad H=\begin{bmatrix}\rho w\\ \rho uw\\ \rho vw\\ \rho w^2+p\\ w\rho\theta\end{bmatrix} \tag{36}$$

In the above relations, ρ is the density of fluid, u is the velocity component in the x-direction, v is the velocity component in the y-direction, w is the velocity component in the z-direction, p is the pressure and θ is the potential temperature. The Euler equations can be written in a quasi-linear form as:

$$U_t+A(U)U_x+B(U)U_y+C(U)U_z=0 \tag{37}$$

The subscripts t, x, y and z denote the derivatives in time and space. $A(U)$, $B(U)$ and $C(U)$ are the coefficient Jacobian matrices. The system in Eq. (37) is hyperbolic, if the matrices $A(U)$, $B(U)$, and $C(U)$ can be diagonalized and have real eigenvalues. $A(U)$, e.g., can be written as:

$$A(U) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} & \frac{\partial f_1}{\partial u_4} & \frac{\partial f_1}{\partial u_5} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} & \frac{\partial f_2}{\partial u_4} & \frac{\partial f_2}{\partial u_5} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial u_3} & \frac{\partial f_3}{\partial u_4} & \frac{\partial f_3}{\partial u_5} \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} & \frac{\partial f_4}{\partial u_3} & \frac{\partial f_4}{\partial u_4} & \frac{\partial f_4}{\partial u_5} \\ \frac{\partial f_5}{\partial u_1} & \frac{\partial f_5}{\partial u_2} & \frac{\partial f_5}{\partial u_3} & \frac{\partial f_5}{\partial u_4} & \frac{\partial f_5}{\partial u_5} \end{bmatrix}, \tag{38}$$

where,

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho\theta \end{bmatrix} \quad and, \quad F(U) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} u_2 \\ \frac{u_2^2}{u_1} + C_o u_5^\gamma \\ \frac{u_2 u_3}{u_1} \\ \frac{u_2 u_4}{u_1} \\ \frac{u_2 u_5}{u_1} \end{bmatrix}. \tag{39}$$

The elements of the matrix in Eq. (38) can be found after some algebraic manipulations, e.g.,

$$\frac{\partial f_2}{\partial u_1} = \frac{\partial}{\partial u_1}\left[\frac{u_2^2}{u_1} + C_o u_5^\gamma\right] = -\frac{u_2^2}{u_1^2} = \left[-\frac{\rho u \rho u}{\rho\rho}\right] = -u^2 \tag{40.1}$$

$$\frac{\partial f_2}{\partial u_2} = \frac{\partial}{\partial u_2}\left[\frac{u_2^2}{u_1} + C_o u_5^\gamma\right] = \frac{2u_2}{u_1} = \frac{2\rho u}{\rho} = 2u \tag{40.2}$$

$$\frac{\partial f_2}{\partial u_3} = \frac{\partial}{\partial u_3}\left[\frac{u_2^2}{u_1} + C_o u_5^\gamma\right] = 0 \tag{40.3}$$

$$\frac{\partial f_2}{\partial u_4} = \frac{\partial}{\partial u_4}\left[\frac{u_2^2}{u_1} + C_o u_5^\gamma\right] = 0 \tag{40.4}$$

$$\frac{\partial f_2}{\partial u_5} = \frac{\partial}{\partial u_5}\left[\frac{u_2^2}{u_1} + C_o u_5^\gamma\right] = \gamma C_o u_5^{\gamma-1} = \left[\frac{\gamma C_o (\rho\theta)^\gamma}{\rho\theta}\right] = \frac{\gamma p}{\rho\theta} = \frac{a^2}{\theta} \tag{40.5}$$

and,

$$\frac{\partial f_3}{\partial u_1} = \frac{\partial}{\partial u_1}\left[\frac{u_2 u_3}{u_1}\right] = -\frac{u_2 u_3}{u_1^2} = -\frac{\rho u \rho v}{\rho\rho} = -uv \tag{40.6}$$

$$\frac{\partial f_3}{\partial u_2} = \frac{\partial}{\partial u_2}\left[\frac{u_2 u_3}{u_1}\right] = \frac{u_3}{u_1} = \frac{\rho v}{\rho} = v \tag{40.7}$$

$$\frac{\partial f_3}{\partial u_3} = \frac{\partial}{\partial u_3}\left[\frac{u_2 u_3}{u_1}\right] = \frac{u_2}{u_1} = \frac{\rho u}{\rho} = u \tag{40.8}$$

$$\frac{\partial f_3}{\partial u_4} = \frac{\partial}{\partial u_4}\left[\frac{u_2 u_3}{u_1}\right] = 0 \tag{40.9}$$

$$\frac{\partial f_3}{\partial u_5} = \frac{\partial}{\partial u_5}\left[\frac{u_2 u_3}{u_1}\right] = 0 \tag{40.10}$$

where, a is the speed of sound. The remaining terms in Eq. (38) can be found in a similar manner. The matrix in Eq. (38) simplifies to:

$$A(U) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -u^2 & 2u & 0 & 0 & a^2/\theta \\ -uv & v & u & 0 & 0 \\ -uw & w & 0 & u & 0 \\ -u\theta & \theta & 0 & 0 & u \end{bmatrix} \tag{41}$$

The eigenvalues and eigenvectors corresponding to the Jacobian matrix *A(U)* in Eq. (41) are:

$$u-a: \begin{bmatrix} 1 \\ u-a \\ v \\ w \\ \theta \end{bmatrix} \quad u: \begin{bmatrix} 1 \\ u \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad u: \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad u: \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad u+a: \begin{bmatrix} 1 \\ u+a \\ v \\ w \\ \theta \end{bmatrix}. \tag{42}$$

The matrix of right eigenvectors can now be written as:

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ u-a & u & 0 & 0 & u+a \\ v & 0 & 1 & 0 & v \\ w & 0 & 0 & 1 & w \\ \theta & 0 & 0 & 0 & \theta \end{bmatrix}. \tag{43}$$

The inverse of the matrix in Eq. (43) is given by:

$$R^{-1} = \begin{bmatrix} \frac{u}{2a} & -\frac{1}{2a} & 0 & 0 & \frac{1}{2\theta} \\ 1 & 0 & 0 & 0 & -\frac{1}{\theta} \\ 0 & 0 & 1 & 0 & -\frac{v}{\theta} \\ 0 & 0 & 0 & 1 & -\frac{w}{\theta} \\ -\frac{u}{2a} & \frac{1}{2a} & 0 & 0 & \frac{1}{2\theta} \end{bmatrix}. \tag{44}$$

The eigenvalues and eigenvectors corresponding to *B(U)* and *C(U)* in Eq. (37) can be found in a similar manner. Once the complete wave structure is known, the algorithm described in this paper can be extended to the third spatial dimension.

Appendix B

The focus of this paper has been on the simulation of Euler equations. It is, however, also necessary to gauge the performance of the proposed scheme when the diffusion operator is included in the solution. The actual Straka non-linear density current benchmark (Straka et al. 1993) is described in this Appendix for the sake of completeness.

The domain in this case was bounded within [-20.0km:20.0km] x [0:6.4km]. Please note that the outflow boundaries in the lateral do not need to be placed too far away, since, the problem of wave reflection at the farfield boundaries is minimal for Godunov-type schemes. The mesh resolution was set to 50m in both the *x*- and *y*-directions. Farfield/outflow boundary conditions were used in the lateral and the top and bottom boundaries were set to solid walls. The initial conditions were defined by taking the following steps:

1. The domain was initialized for a neutral atmosphere by setting the potential temperature at 300 K.
2. The initial *u*-velocity and the *v*-velocity were set to zero.
3. The temperature profile was defined using the following relation:

$$T = 300 - \frac{yg}{c_p} \tag{45}$$

4. Given the potential temperature field, the Exner pressure, Π, was calculated for the entire domain:

$$\frac{\partial \Pi}{\partial y} = \frac{g}{\overline{\theta}} \tag{46}$$

5. Pressure was calculated from the Exner pressure:

$$\Pi = \left(\frac{p}{p_0}\right)^{R_d / c_p} \tag{47}$$

6. A cold bubble was initialized by adding a perturbation in the temperature field using the following relation:

$$\Delta T = \begin{cases} 0.0 & if \;\; L > 1.0, \\ -15.0\left[\frac{\cos(\pi L) + 1.0}{2}\right] & if \;\; L \le 1.0 \end{cases} \tag{48}$$

where,

$$L = \sqrt{\left(\frac{x - x_c}{x_r}\right)^2 + \left(\frac{y - y_c}{y_r}\right)^2} \tag{49}$$

where, x_c = 0.0km, y_c = 3.0km, x_r = 4.0km and y_r = 2.0km. Please note that in the earlier density current simulations the perturbation was added to the potential temperature field.

7. Given the temperature field, potential temperature was re-defined for the entire domain using Eq. (3).
8. Density was initialized using the gas law:

$$p = \rho R_d T \tag{50}$$

The diffusion was added into the solution using a fractional-step method. A constant eddy viscosity/diffusivity ($K_m = K_h = K = 75$ m^2/s) was used for both the momentum and potential temperature fields. The simulation was run for time = 900s. The potential temperature perturbation (K), *u*-velocity (m/s) and the *v*-velocity (m/s) fields at time = 900s are shown in Figure 10. The figures show a portion of the actual computational domain from approximately 0 to 16000m in the *x*-direction and 0 to 6000m in the *y*-direction. The simulation results shown in Figure 10 are in good agreement with the Straka benchmark, both qualitatively and quantitatively (see Table 1). In Table 1, REFC refers to the fully-compressible reference solution, REFS is the fully compressible reference solution on a staggered grid and REFQ is the reference solution using a quasi-compressible model. All reference solutions (REFC, REFS and REFQ) reported in Straka et al. (1993), were on grids with uniform resolution of 25m. The statistics reported in Table 1 are for the right half of the computational domain (shown in Figure 10). The location of the front for the *fwave* solution is in terms of potential temperature and is given by the *x*-coordinate of the cell in the lowest level with a potential temperature of 299.99 K. The Godunov-*fwave* solution also compares well with other reported solutions of the benchmark (e.g., see Janjic et al. 2001; Xue et al. 2000).

Table 1: Comparison of the *fwave*-Godunov results with the Straka Density Current Benchmarks

Variable	*fwave*	REFC	REFS	REFQ
p'_{max} (mb)	1.26	2.87	2.49	1.74
p'_{min} (mb)	-6.27	-5.14	-5.55	-5.21
θ'_{max} (K)	8.92E-03	0.00	0.00	0.00
θ'_{min} (K)	-9.82	-9.77	-9.77	-10.00
u_{max} (m/s)	34.44	36.46	35.02	34.72
u_{min} (m/s)	-15.74	-15.19	-16.32	-15.31
v_{max} (m/s)	13.62	12.93	13.28	13.04
v_{min} (m/s)	-16.36	-15.95	-16.11	-16.89
front location (m)	15525.00	15537.44	N/A	15509.17

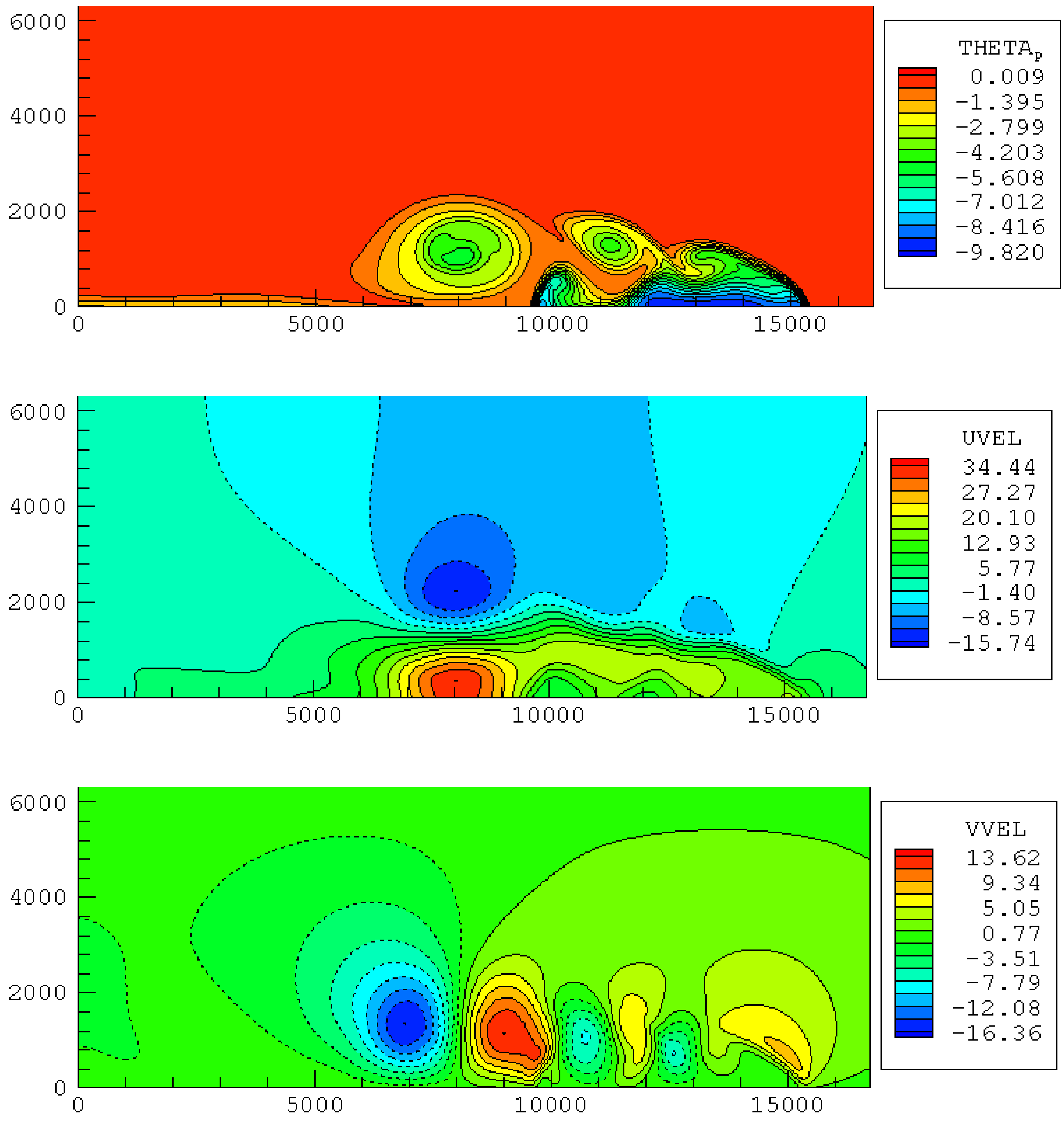

Figure 10. Non-Linear Density Current Benchmark. Potential temperature perturbation (K) – top, *u*-velocity (m/s) – middle, and *v*-velocity (m/s) – bottom. Time = 900s. Dashed lines indicate negative values.

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